Accurate and fast approximations of moment-generating functions and their inversion for log-normal and similar distributions*

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Abstract

A general approximation formula for the moment-generating function of random variables with broadly distributed logarithms is derived using a saddle-point expansion of the defining integral. As a special case, a simple and accurate approximation formula for the moment-generating function of log-normal distributions is obtained. For 4.3 dB spread it reaches 0.5% accuracy and improves rapidly as the spread increases. Exact expressions are derived to obtain, from moment-generating functions of log-normals and related random variables, the moments of their logarithms and the expectation values of their rth powers ($-\infty < r < 1$). Furthermore, an accurate inversion formula that estimates a log-normal-like distribution from the moment generating function and its derivatives is presented. It is conjectured that the formula converges to the exact distribution as higher derivatives are included.

Keywords: Log-normal distributions, Higher order statistics, Inverse problems, Cochannel interference

1 Introduction

Interest in the properties of sums and mixtures of log-normal or similar distributions arises not only from problems of co-channel interference [1, 2, 3, 4, 5, 6, 7] but also in the context of transmission gains of ultra-wideband signals [8], tunnel junctions [9], MOSFET integrated circuits [9], and many other applications.

Sums of log-normals can be described either indirectly in terms of characteristic functions [10] or moment-generating functions [7], or directly in terms of the moments of the sums [3] or their distribution [6, 11, 12]. These characterizations can then either be used directly or to obtain approximations of the distributions of the sums by matching them with other log-normal variables. Different approaches to match the distributions have been proposed [1, 7, 13, 14] and compared [4, 7, 15]. Closed-form expressions for the log-normal characteristic or moment-generating functions do not seem to be available, but efficient methods to numerically evaluate the defining integrals directly [7, 10] or after an elegant transformation [10] have been developed. Approximate expressions for the maxima of distributions of log-normal sums [9] and analytic bounds for the distribution [6] have also been derived.

Particularities of the results reported here are: (1) Similar as in [6], the theory is derived for arbitrary random variables that share the property of log-normals that the logarithm of the variable has a broad, smooth distribution. This is the most natural way to take the fact into account that the sum of log-normals is approximately but not quite log-normal. It will also allow the stochastic analysis to relate better to field data. (2) Exact formulae to obtain the moments of the logarithm of a random variable from its generating function are derived. (3) An inversion formula for the moment-generating function is provided which directly allows the computation of quantiles for sums and mixtures of log-normals. (4) Approximations reach, with minimal computational effort, more than $5 \cdot 10^{-3}$ accuracy at a 4.3 dB spread of the log-normal, and improve quickly at higher spreads. The inversion formula might even yield arbitrarily high precision as the approximation is taken to higher orders.

2 General Theory

2.1 Notation

Define the random variable X as $X = x_0 \exp(\sigma Y)$ where x_0 and σ are constants and the random variable Y is distributed as $P(Y < y) = f(y)$. The probability density is given by $f'(y)$. A log-normal distribution for X is obtained with $f(y) = (1+\mathrm{erf}(y/\sqrt{2}))/2$, i.e. $f'(y) = (2\pi)^{-1/2} \exp(-y^2/2)$, but most considerations here hold for other distributions $f(y)$ as well. If the variance of Y is one, then $\mathrm{std} \ln X = \sigma$, that is, X fluctuates by $(10 \log_{10} e)\, \sigma\, \mathrm{dB} = 4.3\, \sigma\, \mathrm{dB}$. Otherwise, σ is simply a scale parameter.

The moment-generating function $M(t)$ of X is defined as

$$M(t) = \mathsf{E}\left[\exp(t\, X)\right] = \mathsf{E}\left[\exp(t\, x_0 e^{\sigma Y})\right], \tag{1}$$

with $\mathsf{E}[\cdot]$ denoting expectation values. In the analysis below $M(\cdot)$ it often expressed in terms of another function $F(\cdot)$ defined by a change of variables

$$F(y) = M(-e^{-\sigma y}/x_0), \quad M(-t) = F\left(-\ln(t\, x_0)/\sigma\right), \tag{2}$$

for all $y, t > 0$. The theory applies only to the moment-generating function for negative arguments. But this appears to be sufficient to recover all properties of the underlying distribution, assuming X is of the form given above.

2.2 Saddle-point expansion of the moment-generating function

Based on the last expression in Eq. (1), the moment-generating function is given by

$$M(-t) = \int_{-\infty}^{\infty} \exp\left(-t x_0 e^{\sigma y}\right) f'(y)\, dy. \tag{3}$$

The salient point to note now is that $\exp(-tx_0 e^{\sigma y})$ converges to one for $y \to -\infty$ and to zero for $y \to \infty$. At $y = y_0 := -\ln(t\,x_0)/\sigma$ the slope of $\exp(-tx_0 e^{\sigma y})$ is steepest. The larger σ, the steeper is the slope and the sharper the transition from one to zero. Integrating by parts yields

$$M(-t) = F(y_0) = \int_{-\infty}^{\infty} \sigma e^{\sigma(y-y_0)} \exp\left(-e^{\sigma(y-y_0)}\right) f(y)\, dy, \tag{4}$$

with the integrand now localized near y_0. The integral can be approximated by a saddle-point expansion (see Appendix A for details): Expanding $f(y)$ in a Taylor polynomial of order m and exchanging the order of sum and integration yields

$$F(y_0) = \sum_{n=0}^{m} \frac{a_n}{(-\sigma)^n} f^{(n)}(y_0) + \mathcal{O}\left(\frac{1}{\sigma^{m+1}}\right) \tag{5}$$

with $f^{(n)}$ denoting the nth derivative of f. The coefficients $a_0 = 1$, $a_1 = 0.577$, $a_2 = 0.990$, $a_3 = 0.907$, ... approach one for large n. For accurate values, set $a_n = (1 - 2^{-n-1} c_n)$, with the c_n given in Tab. 1. Using symbolic algebra software, exact expressions for the a_i can also be obtained. In particular, a_1 equals Euler's constant γ_E.

Farley's approximation [1, 4, 6] of the distribution of log-normal sums is based on the intuition that, if the spread of the addends is large, the sum is determined by the largest contribution alone. Based on Eq. (5), this intuition can be shown to be correct to first order in $1/\sigma$. First observe that, up to an error of order $\mathcal{O}(\sigma^{-2})$, the moment-generating function of X is related to the cumulative distribution function of Y by a simple change of variables. By Eq. (5)

$$f(y - \gamma_E/\sigma) = f(y) - \gamma_E f'(y)/\sigma + \mathcal{O}(\sigma^{-2}) = F(y) + \mathcal{O}(\sigma^{-2}). \tag{6}$$

Now, let $X_3 = X_1 + X_2$, and define Y_1, Y_2, Y_3 by $Y_n = \ln(X_n/x_0)/\sigma$. Denote the cumulative distribution function of Y_n by $f_n(y)$ $(n = 1, 2, 3)$. The probability that $X_n < x$ is $f_n(\ln(x/x_0)/\sigma)$. The probability that $\max(X_1, X_2) < x$ is the probability that $X_1 < x$ and $X_2 < x$. Thus, intuitively one would assume

$$f_3(\ln(x/x_0)/\sigma) = f_1(\ln(x/x_0)/\sigma)\, f_2(\ln(x/x_0)/\sigma) \tag{7}$$

for any $x > 0$. By applying first Eq. (6) and then Eq. (2), this goes over into

$$M_3(-e^{-\gamma_E}/x) = M_1(-e^{-\gamma_E}/x) M_2(-e^{-\gamma_E}/x). \tag{8}$$

But this is a well known fact: the moment-generating function of the sum of two variables is the product of their generating functions.

2.3 Re-summation

Approximation (5) can be improved further by noting that it implies

$$\tilde{F}(y_0) := F(y_0) + \frac{1}{\sigma} F'(y_0) = \sum_{n=0}^{m} \frac{b_n}{(-\sigma)^n} f^{(n)}(y_0) + \mathcal{O}\left(\frac{1}{\sigma^{m+1}}\right), \tag{9}$$

with $b_n = a_n - a_{n-1} = 2^{-n}(c_{n-1} - c_n/2)$ for $n \geq 1$ and $b_0 = 1$. The coefficients in the sum (9) fall off much faster then those in (5). The function $F(y)$ can be recovered from $\tilde{F}(y)$ by solving the ODE (9) as

$$F(y) = e^{-\sigma y} \int_{-\infty}^{y} e^{\sigma z} \tilde{F}(z)\, dz. \tag{10}$$

Putting the right hand side of Eq. (9) into (10), integrating by parts to replace $\sigma \int_{-\infty}^{y} e^{\sigma z} f(z)\, dz = e^{\sigma y} f(y) - \int_{-\infty}^{y} e^{\sigma z} f'(z)\, dz$ and (for $n > 1$) $\sigma \int_{-\infty}^{y} e^{\sigma z} f^{(n)}(z)\, dz = -e^{\sigma y} \sum_{k=1}^{n-1} (-\sigma)^k f^{(n-k)}(y) - (-\sigma)^n \int_{-\infty}^{y} e^{\sigma z} f'(z)\, dz$, and then re-arranging sums, one obtains

$$F(y) = f(y) - a_m \int_{-\infty}^{y} e^{\sigma(z-y)} f'(z) dz + \sum_{s=1}^{m-1} \frac{a_s - a_m}{(-\sigma)^s} f^{(s)}(y) + \mathcal{O}\left(\frac{1}{\sigma^m}\right). \tag{11}$$

The coefficients in the sum again decay as 2^{-m}. It must be stressed that (5), (9) and (11) are asymptotic formulae. Their accuracy increases always with increasing σ, but at fixed σ with increasing m only up to a certain value. A theoretical argument given in Appendix A suggests that $m = 6$ is often a good choice for Eq. (11). As another general rule, the approximation error can be estimated by the difference it would make if m was incremented by one. This should help finding the best value of m for a particular problem. Numerical examples are provided below.

3 Particular Distributions

3.1 Log-normal distribution

If Y follows a standard-normal distribution, i.e., $f(y) = (1 + \mathrm{erf}(y/\sqrt{2}))/2$, the integral in Eq. (11) can be evaluated analytically. Using the well-known formula for the derivatives of the error function $\mathrm{erf}^{(n)}(z) = 2\,\pi^{-1/2}(-1)^{n-1} H_{n-1}(z) \exp(-z^2)$ with $H_n(z)$ denoting the Hermite polynomials, one obtains

$$F(y) = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{y}{\sqrt{2}}\right) - \frac{a_m}{2} \exp\left(\frac{\sigma^2}{2} - \sigma y\right) \mathrm{erfc}\left(\frac{\sigma - y}{\sqrt{2}}\right) + \exp\left(-\frac{y^2}{2}\right) \sum_{s=1}^{m-1} \frac{a_m - a_s}{\sqrt{\pi}\left(\sqrt{2}\sigma\right)^s} H_{s-1}\left(\frac{y}{\sqrt{2}}\right) + \mathcal{O}\left(\frac{1}{\sigma^m}\right), \tag{12}$$

with $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$ as usual. The formula is easily evaluated numerically. Requiring only evaluation of four transcendental functions and, for fixed σ, one $(m-2)$th order polynomial, it is also fast.

The approximation error of Eq. (12) was obtained by comparison with a high-precision evaluation of the integral in Eq. (4). For the case with $\sigma = 1$ (i.e. 4.3 dB) and $m = 6$, displayed in Fig. 1, the absolute error always remains below $5 \cdot 10^{-3}$. For $\sigma = 2$ (8.7 dB) it is at most $8 \cdot 10^{-5}$ with $m = 6$, and can be reduced further by another order of magnitude by increasing m to about 15. For larger σ the error will be even smaller.

Another point to note is that the optimal order m depends on the range of y considered. If only the upper tail of the distribution—and thus of $F(y)$—is of interest, larger values of m can be used, implying that a_m converges to one.

3.2 Downward log-χ^2 distribution

As another example, consider the case where $-Y$ follows a χ^2 distribution with r degrees of freedom. For intermediate r, the χ^2 distribution becomes a normal distribution with a skew. This might sometimes describe data better then a strictly normally distributed Y.

By definition, $f(y) = 1 - \gamma(r/2, -y/2)/\Gamma(r/2)$ and $f'(y) = 2^{-r/2} e^{y/2} (-y)^{r/2-1}/\Gamma(r/2)$ for $y \leq 0$ and $f(y) = 1$ for $y \geq 0$, where $\gamma(a, x) = \int_0^x e^{-t} t^{a-1} dt$ is the incomplete γ function and $\Gamma(a) = \gamma(a, \infty)$ the usual Γ function. Efficient algorithms to compute $\gamma(a, x)$ or $\gamma(a, x)/\Gamma(a)$ are available (e.g., [16]). Putting this $f(y)$ into Eq. (11) yields for $y < 0$

$$
\begin{aligned}
F(y) =& 1-\frac{\gamma(r/2,-y/2)}{\Gamma(r/2)}-\frac{a_m e^{-\sigma y}}{(1+2\sigma)^{r/2}}\left(1-\frac{\gamma(r/2,-y/2-\sigma y)}{\Gamma(r/2)}\right)\\
&+(-y/2)^{r/2-1}e^{y/2}\sum_{q=0}^{m-2}\frac{(2/y)^q}{\Gamma(r/2-q)}\sum_{s=q+1}^{m-1}\binom{s-1}{q}\frac{a_s-a_m}{(-2\sigma)^s}\\
&+\mathcal{O}\left(\frac{1}{\sigma^m}\right),
\end{aligned}
\tag{13}
$$

where terms in the sum are dropped when $r/2-q$ is a negative integer and $\Gamma(r/2-q)$ has a pole. For $y \geq 0$ one obtains $F(y) = 1 - a_m e^{-\sigma y}(1+2\sigma)^{-r/2}$.

Near $y = 0$, the non-analyticity of $f(y)$ at $y = 0$ can cause problems, in particular for odd or non-integer $r < 2m-2$. Otherwise, the accuracy of formula (13) is similar to that of (12).

3.3 A general numerical method

In situations where an analytic evaluation of the integral in the general formula (11) is difficult, the most efficient way to obtain $F(y)$ might be to integrate the simple differential equation $F' = -\sigma F + \tilde{F}$ numerically with $\tilde{F}$ given by approximation (9). However, this method has not been tested, yet.

4 Moment and inversion formulae

After computing moment-generating functions and manipulating them to obtain the moment-generating functions of sums and mixtures, one generally seeks a characterization of the corresponding distribution. In the following, various tool are provided to characterize a probability distribution by its moment-generating function.

4.1 Raw moments

Ironically, the moment-generating functions obtained using approximation (11) are not well suited to compute higher raw moments of distributions by the standard formula $\mathsf{E}\left[X^k\right] = \lim_{t\to 0^-} M^{(k)}(t)$ if $k > 1$. The reason is that the limits $\sigma \to \infty$ and $t \to 0^-$ are not interchangeable. Some tendency of $d^k M(t)/dt^k$ to approximate $\mathsf{E}\left[X^k\right]$ at intermediately small t can be seen, but other approaches to computing these moments will generally be superior. For log-normal distributions, for example, the moments are known to be given by $\mathsf{E}\left[X^k\right] = x_0^k \exp(k^2\sigma^2/2)$.

For general $F(y)$ obtained from formula (11), the first moment ($k = 1$) is effectively the limit of large y of the integral contained in the formula. Applying $\lim_{t\to 0^-} M'(t)$ to this approximation simply extracts that term, giving the correct result up to a factor a_m.

The situation improves for smaller k. Fractional moments with $0 < k < 1$ can be obtained as

$$
\mathsf{E}\left[X^k\right] = \frac{\sigma x_0^k \int\left[1-F(y)\right]e^{k\sigma y}dy}{|\Gamma(-k)|}, \tag{14}
$$

(henceforth integrals go from $-\infty$ to ∞ if not otherwise indicated) and the kth moment of $1/X$ for any real $k > 0$ as

$$
\mathsf{E}\left[\frac{1}{X^k}\right] = \frac{\sigma \int F(y)e^{-k\sigma y}dy}{x_0^k\Gamma(k)}. \tag{15}
$$

Both formulae are exact for exact $F(y)$. Their derivation is given in Appendix B.

With $F(y)$ given by the expansion for log-normals (12), one finds the numerical relative error of Eqs. (14) and (15) as compared to the exact result $x_0^k \exp(k^2\sigma^2/2)$ to be independent of σ. The reason seems to be that, while absolute errors in $F(y)$ decrease with increasing σ, these remaining errors are amplified by the factors $e^{\pm k\sigma y}$, with the two effects canceling out. For Eq. (14) the relative error decreases with increasing order m and is $4\cdot 10^{-5}$ for $k = 0.5$, $m = 6$. The situation appears to be similar for Eq. (15) with $k < 2$. But for $k \geq 2$, numerics indicate that the error does not decrease beyond a local minimum at $m = 6$, where it is, e.g., $6\cdot 10^{-3}$ for $k = 2$ and $\sigma = 1$.

4.2 Inversion formula

To obtain other characterizations of the distribution function of X from $F(y)$ requires, at least implicitly, an inversion of the saddle-point expansion. With respect to inversion, the direct expansion (5) in terms a_n and the indirect form (9) in terms of b_n are equivalent. Both lead to

$$f(y) = \sum_{n=0}^{m} \frac{\alpha_n}{\sigma^n} F^{(n)}(y) + \mathcal{O}\left(\frac{1}{\sigma^{m+1}}\right) \tag{16}$$

with $\alpha_0 = 1$, $\alpha_1 = \gamma_E = 0.577$, $\alpha_2 = \gamma_E/2 - \pi^2/12 = -0.656$, $\alpha_3 = \gamma_E^3/6 - \gamma_E \pi^2/12 + \zeta(3) = -0.042$, $\alpha_4 = \gamma_E^2(\gamma_E^2 - \pi^2)/24 + \pi^4/1440 + \gamma_E \zeta(3)/3 = 0.167$, etc., where $\zeta(3) = \sum_{l=1}^{\infty} l^{-3}$. The values are obtained by putting Eq. (16) into (5) and enforcing identity at all orders in σ^{-1}. Table. 2 lists the first seven digits of α_0 to α_{20}. As shown in Fig. 2 for α_n up to $n = 40$, the values decay faster than $(n!)^{-0.6}$. Thus, chances are good that the sum in Eq. (16) converges for $m \to \infty$. This might make (16) a suitable complement to other methods to invert moment-generating and characteristic functions numerically (e.g. [17, 18]), applicable in particular for distributions with heavy tails.

As an example, Fig. 3 compares numerical results for the cumulative distribution $f_2(y)$ of the sum of two log-normals with $\sigma = 2$ and $x_0 = 1$, in the first case calculated using Eq. (16), where $F(y)$ is the square of the approximated log-normal generating function given by Eq. (12) (both sums with $m = 6$), and in the second case calculated directly by a precise numerical convolution of a log-normal cumulative distribution and a log-normal density. As shown in Fig. 3, absolute errors do not exceed $5 \cdot 10^{-4}$. Errors decrease quickly with increasing spread. For $\sigma = 1$, on the other hand, the saddle point approximation would be too coarse to yield reliable results for $f_2(y)$ upon inversion.

The error estimate for the saddle-point approximation for $F(y)$ derived in Appendix A is uniform in y. This implies that the method does not guarantee that the relative errors of $f(y)$, $F(y)$, $1-f(y)$ or $1-F(y)$ for extreme values of y are small. In fact, relative errors tend to increase as $|y|$ increases. Yet, the region where the method yields good approximations for log-normal sums will often cover the range of probabilities $1-f(y) \gtrsim 10^{-4}$. For smaller values, Farley's simple approach yields excellent results [4, 6]. As an example, Figure 4 compares complementary distribution functions $P(\sum_i^N X_i > x) = 1 - f_N(\log(x)/\sigma)$ of sums of $N = 2, 6, 20$ identically distributed independent log-normal random variables with $\sigma = 2$ obtained using the method described here (with $m = 9$) with results using Farley's approach and direct simulations.

A question that might arise with Eq. (16) is how to compute the derivatives. Here, automatic symbolic differentiation was used. Another approach might be direct numerical differentiation if a fast, accurate formula for $F(y)$ such as (12) or expressions derived thereof are available. If $F(y)$ is expensive, methods using higher-order spline approximations might be applicable.

4.3 Moments of the logarithm

When computing moments of $\log X$ or equivalently of Y for integer orders k, only terms up to $n = k$ in Eq. (16) contribute. Integrating by parts and using Eq. (16), one obtains, e.g.,

$$\begin{aligned} \mathsf{E}[Y] &= \lim_{l\to\infty} \int_{-\infty}^{l} y f'(y)\, dy \\ &= \lim_{l\to\infty} \left[l\, f(l) - \int_{-\infty}^{l} f(y)\, dy \right] \\ &= \lim_{l\to\infty} \left[l - \int_{-\infty}^{l} F(y)\, dy \right] - \frac{\alpha_1}{\sigma} \\ &= \int_0^{\infty} 1 - F(y) - F(-y)\, dy - \frac{\gamma_E}{\sigma}. \end{aligned} \tag{17}$$

The corresponding result for general positive k reads

$$\mathsf{E}\left[Y^k\right] = \frac{\alpha_k\, k!}{(-\sigma)^k} + \sum_{n=0}^{k-1} \frac{\alpha_n}{(-\sigma)^n} \frac{k!}{(k-n-1)!} I_{k-n-1} \tag{18}$$

with $I_n = \int_0^\infty y^n \, [1 - F(y) - (-1)^n \, F(-y)] \, dy$. By the derivation given in Appendix B, this formula is exact for exact $F(y)$, independent of the convergence of the inversion formula. For $F(y)$ obtained directly using approximation (11) with $m \geq k$, numerics confirm that it is exact as well. Equations (17) and (18) are useful in particular for obtaining log-normal approximations for sums and mixtures of log-normal and similar distributions.

5 Concluding Remarks

It should be emphasized that the methods described here can be used in combination, but also separately. For example, one might simply compare log-normals and sums and mixtures based on their moment generating function [7], without any inversion or computation of moments. One might also compute the moment generating function by other methods that yields better convergence, and only apply the inversion formula or the moment formulae to the generating function of sums and mixtures obtained thereof.

The largest drawback of the method described here is perhaps that approximation (11) for the moment-generating function $F(y)$ does not converge to arbitrary precision as the order m of the approximation increases. This limits the accessible range for $F(y)$ to $\sigma \gtrsim 1$ (4.3 dB spread) and for distribution functions of sums to $\sigma \gtrsim 2$ (8.7 dB spread). On the other hand, since the accuracy improves as σ^{-m}, excellent results are obtained for larger spreads. For small σ it might be the easiest to compute the moment-generating function simply by evaluating the defining integral [7, 10]. Alternatively, the method presented here can be modified to compute the convolution in Eq. (4) and its inversion using Fourier-transform techniques, rather than by an expansion in derivatives. Details of this approach will be described elsewhere.

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Appendices

A Saddle-point expansion

To obtain a saddle point expansion of the integral (4), expand $f(y)$ in a Taylor polynomial of order m at y_0. This yields

$$f(y) = \left[\sum_{n=0}^{m} \frac{(y-y_0)^n}{n!} f^{(n)}(y_0)\right] + \frac{(y-y_0)^{m+1}}{(m+1)!} f^{(m+1)}(\hat{y}), \tag{19}$$

with $\hat{y}$ located between y and y_0. If, as for the normal distribution, all derivatives of f are bounded, $|f^{(m+1)}(\hat{y})|$ is bounded by $C_{m+1} = \max_y |f^{(m+1)}(y)|$ uniformly in y_0. Inserting (19) into (4) yields, after some simple re-arrangements, Eq. (5) with the uniform bound $a_{m+1}C_{m+1}\sigma^{-m-1}$ on the error term and the coefficients

$$a_n = \frac{(-1)^n}{n!} \int_0^\infty e^{-u} (\ln u)^n \, du. \tag{20}$$

For large n, the integral is dominated by a narrow region near $u = 0$, and can be approximated by $\int_{u=0}^{1} (\ln u)^n du = (-1)^n n!$. Hence, the a_n converge to 1.

In principle, an expansion of $f(y)$ at another point $y_0 + \delta/\sigma$ with arbitrary δ would be possible, as well. This would lead to coefficients

$$a_n = \frac{e^\delta (-1)^n}{n!} \int_0^\infty \exp(-ue^\delta)(\ln u)^n \, du. \tag{21}$$

One might attempt to adjust δ such as to minimize the coefficients $b_n = a_n - a_{n-1}$ entering Eq. (9) to obtain better approximations. It turns out that the location of the minimum of b_n depends on n. For $n = 6$, unlike for other n, the minimum is very close to zero ($\delta_{\min} = -0.0036$). Contributions in the approximation (9) for $\tilde{F}(y)$ beyond $n = m = 6$ are, in this sense, not optimal, and dropping them will sometimes improve results.

B Derivation of moment formulae

To obtain Eq. (14) for $0 < k < 1$, observe first that

$$1 - F(y) = \int \left[1 - \exp(-e^{\sigma(z-y)})\right] f'(z) \, dz, \tag{22}$$

by the definition of $F(y)$ and because $\int f'(y)dy = 1$. Thus, with two changes of variables,

$$\begin{aligned}
\int [1 - F(y)]\, e^{k\sigma y} dy &= \int \int \left[1 - \exp(-e^{\sigma(z-y)})\right] e^{k\sigma y} f'(z) \, dz \, dy \\
&= \int \int [1 - \exp(-e^{\sigma w})]\, e^{k\sigma(z-w)} f'(z) \, dz \, dw \\
&= \int [1 - \exp(-e^{\sigma w})]\, e^{-k\sigma w} \, dw \times \int e^{k\sigma z} f'(z) \, dz \\
&= \sigma^{-1} \int_0^\infty [1 - \exp(-u)]\, u^{-k-1} \, du \times \frac{\mathsf{E}\left[X^k\right]}{x_0^k}.
\end{aligned} \tag{23}$$

Evaluation of the remaining integral yields Eq. (14). The derivation of Eq. (15) proceeds along the same lines.

To derive Eq. (18), first introduce the abbreviation $K(y) = \sigma \exp(-e^{\sigma y})e^{\sigma} y$ for the kernel of the integral in Eq. (3). Then, note that, for any polynomial $g(y)$ in y of degree k and a moment generating function $F(y)$,

$$\begin{aligned}\int g(y)F'(y)\,dy &= -\int\int g(y)K'(z-y)f(z)\,dz\,dy \\ &= \int\int g(y)K(z-y)f'(z)\,dz\,dy \\ &= \int G(z)f'(z)\,dz\end{aligned} \tag{24}$$

with $G(-z) = \int K(y-z)g(-y)\,dy$. Since $g(-y)$ and $G(-y)$ are related just as $f(y)$ and $F(y)$ in Eq. (3), the saddle-point expansion (5) applies to them as well. It is exact if $m \geq k$. If one finds a $g(y)$ such that $G(y) = y^k$, then Eq. (24) yields the kth raw moment of Y. Up to the change of sign, this is exactly what inversion formula (16) archives. Thus

$$g(y) = \sum_{n=0}^{k} \frac{\alpha_n}{(-\sigma)^n} \frac{d^n}{dy^n} y^k = \sum_{n=0}^{k} \frac{\alpha_n}{(-\sigma)^n} \frac{k!}{(k-n)!} y^{k-n}. \tag{25}$$

To avoid having to differentiate $F(y)$ to compute the moments, the l.h.s. of Eq. (24) is re-arranged into the form

$$g(0) + \int_0^\infty g'(y)\,[1-F(y)] - g'(-y)F(-y)\,dy \tag{26}$$

by the technique used in Eq. (17). With $g(y)$ given by Eq. (25), this yields Eq. (18).

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n	c_n
0	0
1	1.6911373
2	0.0875520
3	1.4803348
4	0.5847012
5	1.1523156
6	0.8769133
7	1.0235493
8	0.9698848
9	0.9981260
10	0.9919364
11	0.9969766
12	0.9972014
13	0.9983720
14	0.9988419
15	0.9992471
16	0.9994923
17	0.9996627
18	0.9997747
19	0.9998498
20	0.9998998

Table 1: The coefficients $c_n = 2^{n+1}(1 - a_n)$ at seven digits precision.

n	α_n
0	1
1	$+5.772157 \times 10^{-1}$
2	-6.558781×10^{-1}
3	-4.200264×10^{-2}
4	$+1.665386 \times 10^{-1}$
5	-4.219773×10^{-2}
6	-9.621972×10^{-3}
7	$+7.218943 \times 10^{-3}$
8	-1.165168×10^{-3}
9	-2.152417×10^{-4}
10	$+1.280503 \times 10^{-4}$
11	-2.013485×10^{-5}
12	-1.250493×10^{-6}
13	$+1.133027 \times 10^{-6}$
14	-2.056338×10^{-7}
15	$+6.116095 \times 10^{-9}$
16	$+5.002008 \times 10^{-9}$
17	-1.181275×10^{-9}
18	$+1.043427 \times 10^{-10}$
19	$+7.782263 \times 10^{-12}$
20	$-3.696806 \times 10^{-12}$

Table 2: The coefficients α_n at seven digits precision.

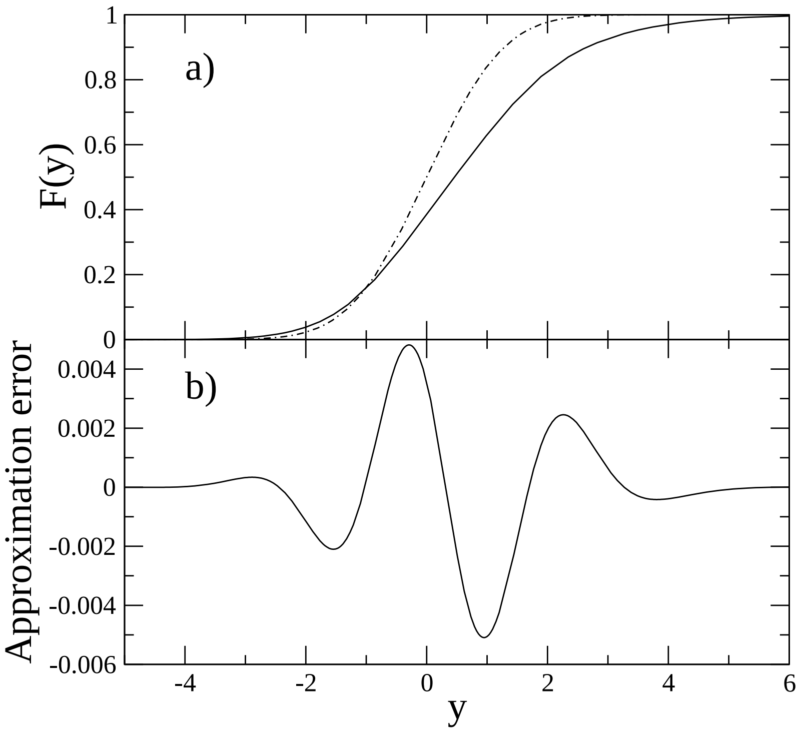

Figure 1: The moment generating function $F(y)$ of a log-normal distribution obtained using approximation (12) with $\sigma = 1$ (4.3 dB spread) and $m = 6$. a) solid line: $F(y)$ (the exact function would be indiscernible); dash dotted line: the cumulative distribution $f(y)$ for comparison. b) The absolute approximation error for $F(y)$.

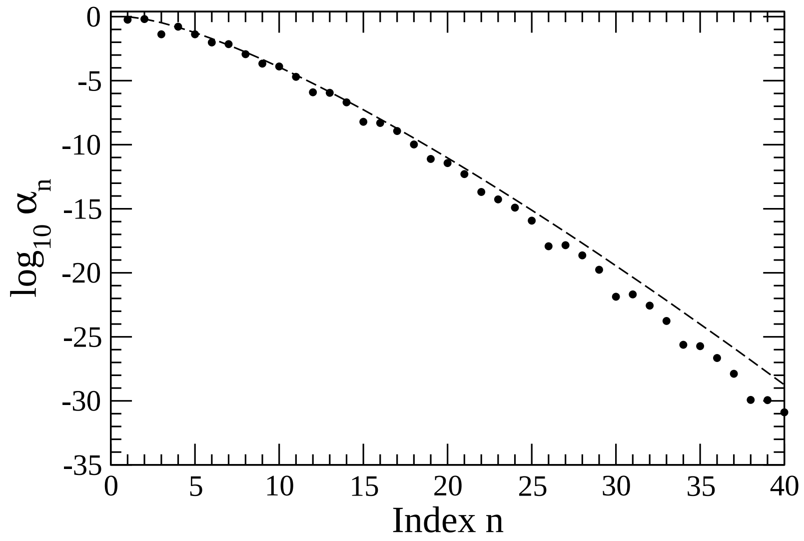

Figure 2: The absolute values of the series coefficients α_n in the inversion formula (16) as a function of n (dots), as compared to $(n!)^{-0.6}$ (dashed line). The fast decay of α_n supports the conjecture that the series (16) converges as $m \to \infty$ for a broad class of moment-generating functions.

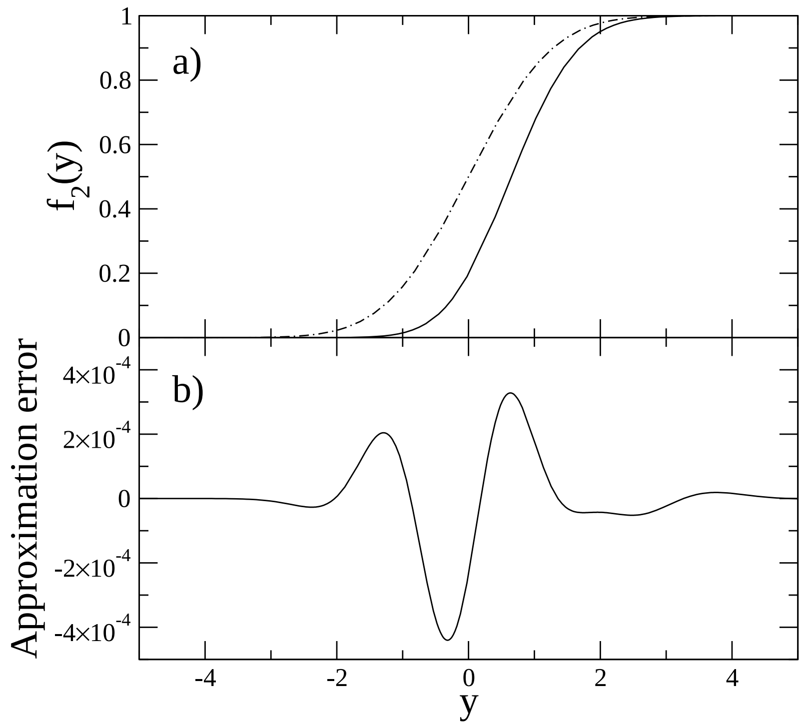

Figure 3: The cumulative distribution function $f_2(y)$ for (the normalized logarithm of) the sum of two log-normal variables with spread $\sigma = 2$ (8.7 dB), obtained using approximations (12) for the moment-generating function and (16) for the inversion of its square, both with $m = 6$. a) solid line: $f_2(y)$ (the numerically exact function would be indiscernible); dash dotted line: the cumulative distribution $f(y)$ of one addend for comparison. b) The approximation error for $f_2(y)$.

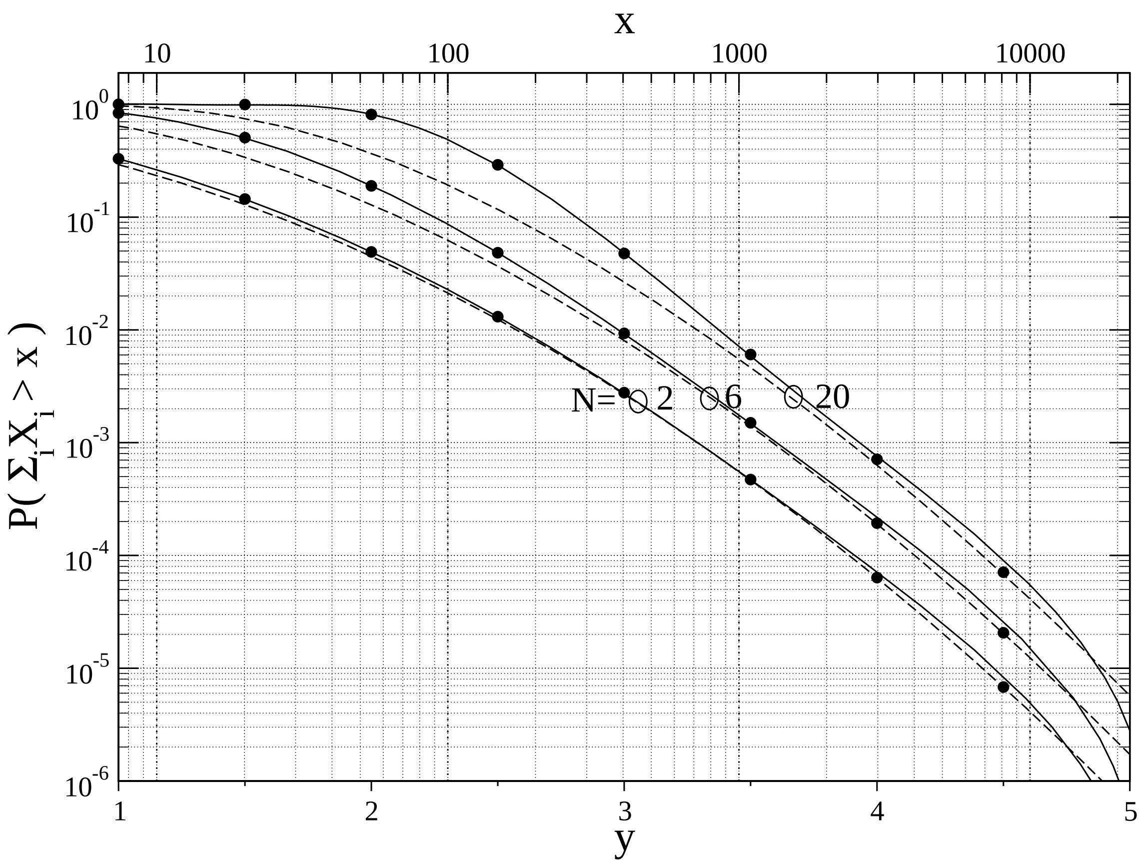

Figure 4: Complementary distribution functions for sums of $N = 2, 6, 20$ independent, log-normal random variables with $x_0 = 1$, and $\sigma = 2$ (i.e., $x = e^{2y}$ for both the sum and its components). Solid line: saddle point expansion (this work, $m = 9$); dashed line: Farley's approach; points: direct simulation.